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Stability of the isotropic fixed point near one dimension

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Abstract. Within the context of the $1+\varepsilon$ expansion and the near-planar interface Hamiltonian, the isotropic fixed point is found to be stable to a wide class of perturbations. In particular, the universality class includes all non-singular forms of anisotropic surface tension.

1. Introduction

Some time ago, the idea of analysing Ising-like systems by an expansion around the lower critical dimension was proposed (Wallace and Zia 1979). This possibility was based on a study of the statistical mechanics of nearly planar interfaces at low temperatures. Since then, many contributions (Diehl *et al* 1980, Forster and Gabriunas 1981a, b, David 1981, Bausch *et al* 1981, Bruce and Wallace 1981a, b, 1983, Lin and Lowe 1983, Schmittmann 1984) have added to the overall understanding of this system, while others (Lowe and Wallace 1980, Kogon and Wallace 1981, Pytte *et al* 1981, Mukamel and Pytte 1982, Lowe 1982, Schmittmann 1982, Grinstein and Ma 1982, 1983, Grinstein 1984) have generalised this approach to study other systems.

Although doubts about the very philosophy of using interfacial fluctuations to describe bulk criticality were harboured in the early stages, explicit criticisms only appeared recently (Huse *et al* 1985, Teitel and Mukamel 1984). The main thrust of these is directed at the possible instability of the isotropic fixed point. Though they argued that the neglect of overhangs spoils rotational invariance, none of the authors demonstrated mathematically how the isotropic fixed point is unstable against overhangs. The beautiful model of Huse *et al* (1985) is unfortunately not a strong witness to support their case, for reasons we will detail in the last section. Certainly, there is no evidence of instability of the isotropic fixed point within the framework of the ε expansion. Nor is there strong evidence otherwise! To date, only one perturbation has been studied (Lin and Lowe 1983) at the one loop order, and it is irrelevant. Since this perturbation corresponds to the mean curvature and is *naively* irrelevant the stability of the isotropic fixed point was not severely 'tested'. On the other hand, there are *infinitely* many perturbations which are naively marginal (since ∇f is dimensionless). Their effects on the isotropic fixed point were not fully understood (Zia and Lowe 1980). Thus, questions concerning the stability properties remain largely unanswered.

The main purpose of this paper is a study of the naively marginal perturbations.

A systematic classification of these operators, in terms of generalised spherical harmonic functions of the normal vector of the interface, emerges. At the one loop level, we find that *all are irrelevant*, except the following three cases. One (finite) set remains marginal, while another, corresponding to the change of temperature, is relevant. The last case, though formally relevant, is not associated with bona fide operators. The possible physical origins of these irrelevant/marginal operators may be anisotropy in the surface tension. If this anisotropy is a non-singular function of angles, then its effects on the isotropic fixed point are not relevant. After a brief review of the methods used to obtain the isotropic fixed point, we provide some details of the analysis of insertions at the one loop level. Finally, we discuss the physics associated with these operators and conclude with some observations.

2. Interfacial fluctuations in $1+\varepsilon$ dimensions

Here, we present a brief review of the framework within which the isotropic fixed point is studied. For details, see Wallace (1984) and references therein.

Consider a bulk system with two-phase coexistence in d-dimensional Euclidean space (with coordinates $\{x_i, z\}$, where $i=1,\ldots,d-1$). Let an interfacial configuration between the phases be specified by $z=f(x)$. To this configuration, we associate an energy functional (Mandelstam 1913, Buff *et al* 1965) which is simply σ times the area of the interface, where σ is the surface tension (surface energy density). To study the statistical mechanics of such a system, we average over all $f(x)$ with the Boltzmann factor $\exp\{-H[f]\}$. Absorbing both σ and the Boltzmann constant into the scale of temperature, we begin simply with the effective Hamiltonian:

$$H[f]=\int \mathrm{d}^{d-1}x(\sqrt{g})/T \qquad (1)$$

where $g\equiv 1+(\nabla f)^2$ and T is the temperature. Parenthetically, note that (1) may be derived from a bulk Hamiltonian in the sense of a low T limit (Diehl *et al* 1980, Lowe 1982, Zia 1985). The physics is then contained in the correlation functions of f and their derivatives.

Before we discuss the field theoretic approach to this problem, some remarks concerning the symmetry properties of H are in order. Manifestly, H is symmetric under transformations of a $(d-1)$-dimensional Euclidean group. The bulk Hamiltonian from which H is derived, however, is symmetric under the full d-dimensional group. The difference lies in the explicit breaking of the full symmetry by $f(x)$. In particular, under translations in z, f changes in the same fashion; while under a z-x_i rotation by an infinitesimal angle ϑ_i, it transforms according to

$$f(y)\to f(x)+\vartheta_i\{x_i+f\nabla_i f\}. \qquad (2)$$

Now, it is straightforward to check that H is invariant under these transformations since $\sqrt{g}$ changes by a total divergence. Thus, the full symmetry is non-linearly realised in H. In the same spirit as the non-linear sigma model (Polyakov 1975, Brezin and Zinn-Justin 1976a, b) for Heisenberg spin systems, our interfacial Hamiltonian contains f as the soft Goldstone modes associated with a spontaneous breaking of a continuous symmetry.

The analysis now proceeds along well known lines, regarding the $\frac{1}{2}(\nabla f)^2$ part of H as a free Hamiltonian. The higher order terms

$$U(\nabla f) \equiv \sqrt{g} - 1 - \tfrac{1}{2}\nabla f^2 \tag{3}$$

appearing via $U \equiv \int \mathrm{d}x\, U/T$, are treated as perturbation. In practice, a further quadratic 'mass' term, $\int \mathrm{d}x(m^2 f^2)/T$, must be added for a well defined theory†. The physical origin of this term can be gravity for a liquid–gas system, for example. Mathematically, it is a necessary infrared regulator, leading to the propagator $T/(k^2+m^2)$. From (3), we see that every vertex comes with a factor of $1/T$ so that a systematic expansion in powers of T, corresponding to the loop expansion, emerges. Note that (*a*) T plays the role of the coupling constant and (*b*) it is consistent with the idea that H is expected to be good for small T. When we study the dimensions of various quantities, we find that the simplest choice is to keep ∇f (and $\sqrt{g}$) dimensionless for any d, so that m is an inverse length (capillary length for liquid–gas). Then, T becomes dimensionless for $d=1$. Expecting the theory to be renormalisable for this d, we are led naturally to consider renormalisation group analyses for these systems in an $\varepsilon \equiv d-1$ expansion. Employing dimensional regularisation, we find that all vertex functions Γ can be rendered free of poles in ε, if expressed in terms of the renormalised temperature t and 'mass' m_R. The renormalised vertex functions satisfy the renormalisation group equations. Appearing in these equations is $\beta(t)$, which controls the change of t with respect to a scale change. Since we are interested in $\mathrm{O}(\varepsilon)$ results, it is sufficient to quote $\beta = \varepsilon t - t^2 + \ldots$ Thus we arrive at a 'trivial' fixed point $t=0$ and, for $\varepsilon>0$, a non-trivial one:

$$t_c = \varepsilon + \mathrm{O}(\varepsilon^2). \tag{4}$$

The former represents a system at $T=0$, with a flat interface, breaking the rotational part of the full d-dimensional Euclidean symmetry. The latter, which we call the isotropic fixed point, preserves this symmetry, i.e. a typical configuration associated with this point is a scale invariant fractal interface (David 1981).

Next we study the stability properties of the fixed points. Within the subspace of t alone, the zero temperature fixed point is infrared stable and uninteresting. But the isotropic fixed point is unstable, allowing us to identify it with a system at criticality. (Of course it is unstable against the symmetry breaking field m^2). Apart from one case (Lin and Lowe 1983), the effect of no other interaction on this fixed point is understood. A systematic characterisation of other interactions is their naive dimensions. Since the basic field f has dimension -1 (i.e. momentum^{-1}), perturbations with any power of f without derivatives are expected to be relevant. However, such interactions represent an explicit breaking of the translational part of the full Euclidean symmetry. Simple examples like inhomogeneous magnetic or gravitational fields are known to drive the system away from criticality. Their presence must be controlled to arrive at a critical point. In this paper, we will not be concerned with such fields, except, of course, the infrared regulating 'mass' term.

On the other hand, derivatives of f are translationally invariant (although not rotationally invariant). The lowest dimension object in this class is ∇f, with zero dimension. Thus any power of it is still dimensionless and we are faced with an infinite set of such marginal operators, as should be expected in physical systems. Indeed, even in the continuum limit, homogeneous anisotropic interactions are inherited from

† We now have the usual Gaussian measure for the functional integrals.

the underlying lattice structure in many examples: uniaxial magnets, binary alloys, the Ising model itself, etc. Now, we believe anisotropy to be irrelevant, if the fixed point is to describe bulk criticality. Yet the effect of such interactions on the isotropic fixed point has not been analysed. In the next section, we attempt to remedy this defect with a study, at the lowest order in ε, of this 'most dangerous' set† of perturbations.

3. Perturbations at the one loop level

In this section, we derive in some detail the effects on the isotropic fixed point due to insertions of the naively dimensionless composite operators. That is, in the critical dimension $d=1$, these would be of dimension zero. Now, in the renormalisation group language, an operator with a negative dimension would be classified relevant and would produce crossover behaviour. On the other hand, positive dimension perturbations are considered 'safe' and produce only corrections to scaling, with the leading singularities being universal. Thus, the operators we propose to consider do not seem 'dangerous'. Two observations caution us otherwise, however.

(i) Since we wish to study a non-trivial fixed point, we should expect anomalous dimensions in addition to the naive quantities.

(ii) For $d>1$, even the physical dimension is *negative* $(-\varepsilon)$, so that all the perturbations appear to be relevant. At first sight, the isotropic fixed point seems doomed.

Fortunately, the anomalous dimensions are positive (except in two cases) and no (physically meaningful) perturbation on the isotropic fixed point is relevant.

Since ∇f is dimensionless, the class of 'dimensionless' operators is just integrals of general functions, G, of the local field $\nabla f(x)$:

$$V[f]=v\int \mathrm{d}^{\varepsilon}x\, G(\nabla f). \tag{5}$$

Here v is a arbitrary small constant, with G normalised somehow. To study their effects on the system we follow the standard route (Amit 1984) and consider vertex functions with one insertion of vG at zero momentum. (This is equivalent to studying vertex functions associated with $H+V$ up to all order in T but only to first order in v.) Extra divergences are to be absorbed into v via the renormalisation

$$v=v_R Z_v(t) \tag{6}$$

where Z_v would contain only a simple pole in ε at the one loop order. Of course, this equation is highly symbolic since an insertion of an arbitrary G would not be renormalisable with a simple factor. Instead, Z_v should be regarded as a matrix which displays the mixing of the (infinitely) many operators corresponding to different G. Note that, on dimensional grounds, we do not expect mixing of operators outside this class.

To be specific, we analyse all one loop diagrams (figure 1) with one V and many U vertices. Extracting a pair of ∇f from V for the internal lines, we are left with

$$G_{ij}\equiv\tfrac{1}{2}\partial^2 G/(\partial\nabla_i f\partial\nabla_j f) \tag{7}$$

† The curvature term analysed by Lin and Lowe (1983) consists of one factor of $\nabla^2 f$ and many of ∇f. Having dimension one, it is naively 'safe'.

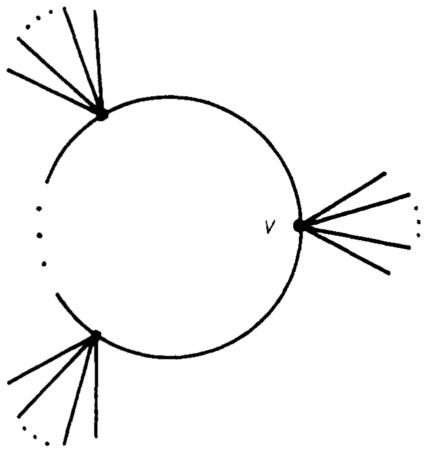

Figure 1. A typical one loop graph with one V insertion.

for the external ones. Together with the momenta coming from contractions, such a vertex contributes a factor $\frac{1}{2}k_iG_{ij}k_j$ to the final integral over the internal momentum k. Similarly, we have $\frac{1}{2}k_iU_{ij}k_j/T$ for each of the other vertices. An internal line comes with $T/(k^2+1)$, where we have set the mass to unity. Explicitly, with all the factors from lth order perturbation theory and contraction combinatorics, we have

$$\int \mathrm{d}^\varepsilon k \sum_l [(-1)^l/l!](\tfrac{1}{2}vTkGk)(\tfrac{1}{2}kUk)^l(k^2+1)^{-l-1}(2^l l!) \tag{8}$$

where the sum extends from 0 to ∞ and the indices i, j are suppressed. The integration should produce a $1/\varepsilon$ factor, so that the divergent contribution is proportional to vT/ε.

Summing the series leads to the integrand $kGk/[(k^2+1)+kUk]$. Change the variables by $k=(\sqrt{J})q$, where the matrix J is defined by

$$J_{ij}^{-1} \equiv \delta_{ij} + U_{ij}. \tag{9}$$

Now, the integral reduces to the standard one (Wallace and Zia 1979, Forster and Gabriunas 1981a, b):

$$(2\pi)^{-\varepsilon}\int \mathrm{d}^\varepsilon q\, q_iq_j/(q^2+1) = -\alpha\delta_{ij}/\varepsilon$$

where $\alpha \equiv (4\pi)^{-\varepsilon/2}\Gamma(1-\frac{1}{2}\varepsilon)$ is absorbed into the renormalised temperature t. Thus, we obtain

$$(-vt/2\varepsilon)\surd[\det J]\,\mathrm{Tr}(JG). \tag{10}$$

It is straightforward to derive $J_{ij} = g_{ij}\surd g$, where $g_{ij} \equiv \delta_{ij} + \nabla_i f \nabla_j f$ is the metric for the interface and g is its determinant. Note that, to lowest order in ε, $\det J$ is just g. The final result for all the one loop graphs is

$$(-vt/2\varepsilon)g(g_{ij}G_{ij}). \tag{11}$$

The interpretation of this simple formula is as follows. It is a certain (local) function of $\nabla f(x)$. In the Γ corresponding to this function, we have a divergence as a result of a single insertion of V.

In general, (11) is not proportional to $G(\nabla f)$ so that the one loop divergent part is not proportional to the tree contribution. However, (11) is linear in G by construction, so that the effect of renormalisation is revealed in a matrix operation. To calculate the anomalous dimensions, we must first diagonalise this matrix and find the eigenoperators. Under renormalisation, the eigenoperators would change by a simple multiplicative factor and are called scaling fields (Wegner 1972).

Thus, we seek those G which satisfy the differential equation

$$(1+\xi^2)\{\delta_{ij}+\xi_i\xi_j\}\partial_i\partial_j G(\xi)=\lambda G(\xi) \tag{12}$$

where λ is an eigenvalue. Although it is feasible to find solutions to the equation in the straightforward way (Zia and Lowe 1980), we discover that, without some appropriate boundary conditions, the problem is ill defined. Some physical content must be added to arrive at these boundary conditions.

For insight, we draw again on the close analogy between the interface problem and the non-linear sigma model of the O(n) symmetric Heisenberg spin systems in $2+\varepsilon$ dimensions (Polyakov 1975, Brezin and Zinn-Justin 1976a, b). In particular, the mixing of an infinite number of operators† also appears in the analysis of that isotropic fixed point (Brezin *et al* 1976a, b). Recognising the importance of the full O(n) symmetry, these authors showed that operators corresponding to the irreducible representations of O(n) are the eigenoperators. This leads us to explore the transformation properties of our V.

Clearly, only the z-x rotation part of the full group needs attention. In some sense, $\sqrt{g}\,dx$ is already an invariant and should be extracted out of G. So we define F by

$$G=F\sqrt{g}. \tag{13}$$

With the full d dimensions in mind, we tailor F so that it depends on ∇f through a bona fide d-dimensional vector. The most natural one is the unit normal to the interface‡: n_μ, $\mu=1,\ldots,d$. Explicitly

$$n_i\equiv\nabla_i f/\sqrt{g} \qquad n_d\equiv-1/\sqrt{g}. \tag{14}$$

Now we expect that, if F is chosen to be a generalised spherical harmonic function of n_μ, it will lead to an eigenoperator. To check, we first express the differential operator in (11) and (12) in terms of derivatives δ_i with respect to n_i instead of that with respect to $\nabla_i f$:

$$\delta_i=(\sqrt{g})g_{ij}\partial/\partial\nabla_j f.$$

Note that $\delta_i n_d$ is not zero but $\sqrt{g}\,n_i$.

Again, keeping terms only to lowest order in ε, we obtain, instead of (11),

$$(-vt/2\varepsilon)\sqrt{g}(h_{ij}\delta_i\delta_j F) \tag{15}$$

where $h_{ij}\equiv\delta_{ij}-n_in_j$. Equation (12) is replaced by an even simpler one:

$$h_{ij}\delta_i\delta_j F=\lambda F. \tag{16}$$

Note that $h_{ij}\delta_i\delta_j n_d\propto\varepsilon$ and $h_{ij}(\delta_i n_\mu)(\delta_j n_\nu)=\delta_{\mu\nu}-n_\mu n_\nu$, so that (symmetric) *traceless* tensors of any rank will solve equation (16). Specifically, with $\lambda=-N(N-1)$, the solutions are

$$F^{N,m}=S^{N,m}_{\mu_1\ldots\mu_N}\{n_{\mu_1}\ldots n_{\mu_N}\} \tag{17}$$

† The physics of the systems being different, the operators considered by Brezin *et al* (1976a, b) are naively relevant as opposed to marginal.

‡ We use Latin indices for the transverse $d-1$ dimensions and Greek ones for the full space.

where m runs over the many traceless S with N indices, a reflection of the familiar degeneracies encountered in ordinary angular momentum. Of course, these objects are the generalised spherical harmonics in d dimensions and can be normalised in the usual way. It is instructive to consider the transformation properties of $V_{N,m} \equiv v_{N,m} \int F^{N,m}\sqrt{g}$ under (2) and verify that they indeed form irreducible representations of the full Euclidean group. In this light, the missing boundary conditions that go with (12) are finiteness of G at $\xi = 0$ and ∞, corresponding to finiteness of F for $\boldsymbol{n}$ parallel and perpendicular to z.

Having found the eigenoperators, we may now write a scalar equation for the renormalisation of $v_{N,m}$:

$$v_{N,m} = (v_{N,m})_R Z_N$$

with

$$Z_N(t) = 1 + \tfrac{1}{2}N(N-1)t/\varepsilon + \ldots. \tag{18}$$

The renormalisation group equations for vertex functions with these insertions will contain the additional beta functions, $\beta_N(t) = \beta(t)\partial_t \ln(Z_N) = \frac{1}{2}N(N-1)t + O(\varepsilon)$. At the isotropic fixed point (4), these reduce to

$$\beta_N(t_c) = \tfrac{1}{2}N(N-1)\varepsilon + O(\varepsilon^2) \tag{19}$$

which are precisely the anomalous dimensions of $V_{N,m}$.

To complete the study of the stability of the isotropic fixed point against perturbations by this class of operators, we must add the naive dimension $-\varepsilon$ to (19):

$$\omega_{N,m} = \tfrac{1}{2}(N^2 - N - 2)\varepsilon + O(\varepsilon^2). \tag{20}$$

Near criticality where $\tau \equiv (T_c - T)$ is small, if the singular part of a thermodynamic quantity of the system were to be represented by $\Gamma[(v_{N,m})_R;\, \tau]$, then (20) implies

$$\Gamma(v_R;\, \tau) = \Gamma(0;\, \tau)[1 + A v_R \tau^{\omega\nu} + O(v^2)] \tag{21}$$

where ν is the usual correlation length exponent and A is a non-universal amplitude. Thus, the sign of $\omega_{N,m}$ determines whether the corresponding interaction is relevant or not.

This analysis shows that the only relevant interactions correspond to $N = 0$ or 1.

The $N = 0$ operator is just H itself, so it represents the changing of temperature T. Indeed, this exponent should be simply $\partial\beta(t_c)/\partial t$, to all orders.

The $N = 1$ operators are not really operators at all; in our framework, $\int n_i\sqrt{g}$ is an (ordinarily neglected) surface term while $\int n_d\sqrt{g}$ is a constant, independent of the fields! That the latter is 'relevant' reflects nothing but the physical dimension of $d^\varepsilon x$.

The special role of these two cases in fact provides a useful check on this and future computations. Beyond these we see that the $N = 2$ interactions are marginal (at this order), while *all the rest are irrelevant.* In the next section, we discuss the physical significance of this analysis and conclude with some remarks.

4. Discussions and conclusion

We have studied the effects of a class of interactions on the isotropic fixed point. The choice of this class is based on the renormalisation group idea that operators of low dimensions are more relevant. Restricting ourselves to those which preserve translational invariance, we considered the ones of lowest dimension. If the underlying bulk system also satisfies rotational invariance, then all such interactions except H itself are absent. Our study is not academic, however, since all Ising-like systems, apart from the liquid–gas case, are *anisotropic*. In the low temperature two-phase regime, where the most excitable modes are the interfacial fluctuations, it should still be possible to use an effective Hamiltonian like H. The most reasonable guess would not be (1) but (Zia 1984)

$$\int \mathrm{d}x[\sigma(\boldsymbol{n})\sqrt{g}]/T \qquad (22)$$

where the anisotropy of the underlying system shows up as an $\boldsymbol{n}$-dependent surface tension $\sigma(\boldsymbol{n})$. Note that, as it stands in (22), the anisotropic surface tension is a dimensionless object, since its physical dimension has been absorbed into the definition of T.

Assuming for the moment that this anisotropic surface tension is a non-singular function of $\boldsymbol{n}$, we may expand it in a series of (d-dimensional) spherical harmonics, i.e. the traceless tensors (17). We next argue that the series contains only terms with even N. The anisotropic surface tension is the excess (free) energy density required to create a planar interface between the two phases. Although it is expected to depend on the orientation, it should not depend on which phase is on which side of the interface. Defining the normal as pointing from one phase to the other, we see that $\sigma(\boldsymbol{n})$ should be an even function of $\boldsymbol{n}$. Since N is simply the number of factors of $\boldsymbol{n}$ in $F^{N,m}$, we need only the even N ones to represent σ. The conclusion is that the isotropic fixed point is *not unstable* against perturbations coming from analytic forms of anisotropy, to order ε. Note that, because the $N=2$ harmonics are marginal, we cannot state that the isotropic fixed point is absolutely stable at this stage. For this reason, a two loop calculation is crucial. Such a calculation should be feasible since (*a*) the more elegant techniques of Forster and Gabriunas (1981a, b) can be generalised and (*b*) the operators (17) are expected to remain eigenoperators at all orders, thanks to their irreducible character.

What about singular forms of the anisotropic surface tension? It is possible for $\sigma(\boldsymbol{n})$ to develop cusps (discontinuities in the first derivative) at certain points when the temperature falls below the roughening one. In that case, σ will have terms linear in $\boldsymbol{n}$, although absolute values appear in order to preserve evenness. Would the $N=1$ interaction appear? Is the isotropic fixed point definitely unstable in such circumstances? The short answer to these questions is no. Mathematically, since $\sigma(\boldsymbol{n})$ is still even, a cusp will be represented by an even N series in much the same way a square wave is represented by a Fourier series. Physically, we can argue that our approach is valid in the sense of an expansion about $d=1$. Now, even for $d=2$, the interface is believed to be rough down to $T=0$. Thus, within the framework of the $1+\varepsilon$ expansion, it is reasonable to restrict our attention to smooth $\sigma(\boldsymbol{n})$. Since roughening transitions are subtle phenomena in $d=3$ typically, we should not be surprised that they are beyond the powers of this formalism.

So far, we have studied only the dimensionless operators. We plan to extend this analysis to ones with higher naive dimension. Physically, these interactions correspond to curvature terms in the interfacial Hamiltonian (Lin and Lowe 1983, Zia 1985) and are of interest in another context. In view of the one loop result for the mean curvature (Lin and Lowe 1983) (no $O(\varepsilon)$ term in the full dimension) to carry out these calculations would be more for the sake of completeness than urgency. Higher curvature terms have even larger naive dimensions and are not expected to become relevant for the isotropic fixed point.

Returning to the criticism of Huse *et al* (1985), which motivated this analysis, we believe that the isotropic fixed point is stable against perturbations associated with anisotropy. In a sense, this is an expected result based on the 'orthodox' view that singular critical behaviour is not sensitive to the underlying lattice structure. So, we would argue that the criticisms of Huse *et al* (1985) are misdirected. The configurations we sum over and the naive measure we use *are rotationally invariant.* Indeed, the non-trivial invariant measure (Bruce and Wallace 1983) $\int \mathrm{d}x \ln[1+(f/R)]$ for fluctuations around a spherical interface of radius R reduces to the one used here as $R \to \infty$. The isotropic fixed point does describe a rotationally invariant system and it is stable† within the context of the ε expansion.

Nevertheless, we should be cautious in taking the $1+\varepsilon$ results blindly up to physical dimensions. The unease felt by many authors concerning the neglect of overhang configurations is justified. In order not to contradict our previous statement, there must be a measure for these configurations which also satisfies rotational invariance. Unfortunately, we do not know how to construct such a measure, let alone prove our expectation. On the other hand, Huse *et al* (1985) have not disproved this conjecture. To complicate the problem, the overhangs will probably contribute at the level of $\exp(-1/\epsilon)$ and so beyond the reach of ε expansions. The lesson we learn is not that the statistical mechanics of near-planar interfaces must be anisotropic by construction. Instead, we must be open to the possibility of non-perturbative contributions which are also isotropic.

Indeed, in the beautiful model of Huse *et al* (1985) there is an isotropic fixed point which splits away from the $d=2$ Ising fixed point by such exponentially small amounts, with the latter being the stable one. However, that isotropic fixed point is not likely to be ours either, for two reasons. Our fixed point is characterised by a singly fractal interface, while theirs contains many interfaces. Another difficulty lies in all calculations carried out in *exactly two* dimensions. There, interface models (with short ranged interactions such as functions of ∇f) can be reduced to free field theories, echoing a similar phenomenon (Brezin and Zinn-Justin 1976a, b) for the O(2) non-linear sigma model. By contrast, our model cannot be reduced to a free theory.

At first sight, such discussions seem to imply that the $1+\varepsilon$ expansion is simply an academic pursuit, of no more than pedagogical value. This point of view is sharpened by the availability of many exact results from the $d=2$ Ising model and the reliability of Borel–Padé estimates from $d=3$ field theories (Nickel 1973, Baker *et al* 1978, Le Guillou and Zinn-Justin 1980). We end, therefore, by offering some practical use of the results from the $1+\varepsilon$ expansion. Not all the universal critical quantities associated with two-dimensional Ising-like systems are susceptible to exact analysis on the original model. One example is critical dynamics, and another is corrections to scaling coming from asymmetry of the two phases (e.g. liquid–gas). Since the isotropic fixed point is expected to be at least exponentially close to the true fixed point near $d=1$, the numerical results can also be trusted to this level. These can in turn be used in an interpolating scheme in conjunction with the results from the expansion around the upper critical dimension four. Although there is no rigorous proof that such a scheme is valid, the expectation is that, at least numerically, the estimate should be better than

† Note the contrast between our situation and that of the $O(n)$ symmetric fixed point which is unstable against all anisotropic interactions.

simple extrapolation from either end. Such a method was used (Bausch *et al* 1981) for the dynamic exponent z in a pure relaxational model with some success. Presumably, it can also be employed for ω_5, the leading exponent related to the asymmetry of the phases. A three loop result from the $(4-d)$ expansion already exists (Zhang and Zia 1982). On the other hand, the leading asymmetric interaction in the low temperature regime is the mean curvature (Fisher and Wortis 1984, Lin and Lowe 1983). But here, only the one loop contribution is known. In this sense, a two loop calculation of this exponent is not just another academic exercise. Beyond these points, we must look toward systems other than ferromagnetic Ising ones for ground breaking applications of the low temperature expansion schemes.

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